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Equioscillation under Nonuniqueness in the Approximation of Continuous Functions

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1. INTRODUCTION

We are concerned with the approximation problem

$$\operatorname*{Minimize}_{g \in M} \| f - g \|,$$

where M is a finite dimensional linear subspace of $C[a, b]$, the space of real valued continuous functions defined on the finite interval $[a, b]$, and where $\| \cdot \|$ denotes the supremum norm. For each $f \in C[a, b]$ we let

$$V_f(M) = \{h : \| f - h \| \leqslant \| f - g \| \qquad \text{for all } g \in M\}.$$

It is well known that $V_f(M)$ is a singleton for each $f \in C[a, b]$, if and only if, M has the *Chebyshev property*:

$$\begin{array}{l} g \in M,\ g \neq 0 \textit{ implies that } g \textit{ has at most} \\ n - 1 \textit{ distinct zeroes in } [a, b], \textit{ where} \\ n = \textit{dimension of } M. \end{array} \tag{C}$$

It is also well known that if M has property (C) then for each $f \in C[a, b]$

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the error function $e = f - g$, where $\{g\} = V_f(M)$, *equioscillates*; i.e., there exist $n + 1$ distinct points $a \leqslant x_1 < \cdots < x_{n+1} \leqslant b$ such that

$$| e(x_i)| = \| e \|, \quad i = 1,..., n + 1, \qquad \text{and} \qquad e(x_i)\, e(x_{i+1}) \leqslant 0, \quad i = 1,..., n.$$

However, if M fails to have property (C), then for a given $f \in C[a, b]$ there may or may not exist a $g \in V_f(M)$ such that the error $e = f - g$ equioscillates. Our main result completely characterizes those M for which equioscillation holds for at least one $g \in V_f(M)$.

THEOREM. *Let M be a linear subspace of $C[a, b]$ of finite dimension n. For each $f \in C[a, b]$ there exists at least one $g \in V_f(M)$ such that the error $e = f - g$ equioscillates if and only if M has the Weak Chebychev property*:

Each $g \in M$ has at most $n - 1$ changes of sign;
i.e., there do not exist points $a \leqslant x_1 < \cdots < x_{n+1} \leqslant b$
such that $g(x_i)\, g(x_{i+1}) < 0$, $i = 1,..., n$. (WC)

The proof of this theorem as well as a corollary are given in Section 2. We note that if M has property (C) then it also has property (WC), but not conversely in general. Alternate formulations of (C) and (WC) are given in Section 2. Examples and concluding remarks appear in Section 3.

2. On Chebyshev and Weak Chebyshev Subspaces

It is well known and easily verified that property (C) is equivalent to each of the following two.

If $\phi_1, ..., \phi_n$ is a basis of M then
$a \leqslant t_1 < \cdots < t_n \leqslant b$, $a \leqslant s_1 < \cdots < s_n \leqslant b$ imply (C-1)
$\det[\phi_i(t_j)] \det[\phi_i(s_j)] > 0$.

Given $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$
there exists a $g \in M$ with $g(a), g(b) \neq 0$ such that (C-2)
$(-1)^{i+1} g(x) > 0$, $x_{i-1} < x < x_i$, $i = 1,..., n$.

To get analogous reformulations of property (WC) we simply replace some strict inequalities by loose ones. Thus, we propose to show that property (WC) is equivalent to each of the following two.

If $\phi_1, ..., \phi_n$ is a basis of M then
$a \leqslant t_1 < \cdots < t_n \leqslant b$, $a \leqslant s_1 < \cdots < s_n \leqslant b$ imply (WC-1)
$\det[\phi_i(t_j)] \det[\phi_i(s_j)] \geqslant 0$.

$$\begin{array}{ll} \text{Given } a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b & \\ \text{there exist a } g \in M,\ g \neq 0, \text{ such that} & \text{(WC-2)} \\ (-1)^{i+1} g(x) \geqslant 0,\ x_{i-1} < x < x_i,\ i = 1,\dots, n. & \end{array}$$

The equivalence of (WC-1) (WC-2) and (WC) can be exhibited in a straightforward manner. However, the proofs can be considerably simplified by the following Lemma which is also used in the proof of the Theorem.

LEMMA. *Suppose that M satisfies property* (WC-1) *and that $\phi_1,\dots, \phi_n$ is a basis for M. For each $\sigma > 0$ and $i = 1,\dots, n$ define the function ψ_i^σ by*

$$\psi_i^\sigma(t) = 1/\sigma\sqrt{2\pi}\int_a^b \phi_i(x)\, e^{-1/2[(t-x)/\sigma]^2}\, dx, \qquad a \leqslant t \leqslant b.$$

Then for each σ the subspace M_σ spanned by ψ_i^σ, $i = 1,\dots, n$, has property (C-1). *Moreover on any subinterval $[a', b']$, $a < a' < b' < b$, $\psi_i^\sigma \to \phi_i$, uniformly, as $\sigma \to 0$, for $i = 1,\dots, n$.*

The proof of the first part of the Lemma can be found in Karlin and Studden ([1], page 15), and the second part is readily verified. We now prove the equivalence of properties (WC), (WC-1) and (WC-2).

(WC-1) $\Rightarrow$ (WC). Suppose M has property (WC-1) and suppose there exists a $g \in M$ which changes sign more than $n - 1$ times. Then by the Lemma we can choose $\sigma > 0$ so as to obtain an $h \in M$ which is sufficiently close to g to have at least n zeroes. Since M has property (C-1) this is a contradiction.

(WC-1) $\Rightarrow$ (WC-2). Let $a = x_0 < x_1 < \cdots x_{n-1} < x_n = b$ be given. If M has property (WC-1) then for each $\sigma > 0$ we let M_σ be as in the Lemma. Since M_σ has property (C-2) we may choose $g_\sigma \in M_\sigma$ such that $\| g_\sigma \| = 1$ and $(-1)^{i+1} g_\sigma(x) > 0$, $x_{i-1} < x < x_i$, $i = 1,\dots, n$. It is readily verified that $g_\sigma \to g \in M$ as $\sigma \to 0$ ($\| g \| = 1$) and that $(-1)^{i+1} g(x) \geqslant 0$, $x_{i-1} < x < x_i$, $i = 1, 2,\dots, n$.

(WC-2) $\Rightarrow$ (WC-1). Let $\phi_1,\dots, \phi_n$ be a basis for M. Let $a < s_1 < \cdots < s_n < b$ be points such that $\det[\phi_i(s_j)] \neq 0$. Define in M (uniquely) the functions u_j, $j = 1,\dots, n$, by $u_j(s_i) = (-1)^{i+1}\delta_{ij}$, $i = 1,\dots, n$. Then $u_1,\dots, u_n$ also form a basis for M. Given k, $1 \leqslant k \leqslant n$, let $x_0 = a$, $x_i = s_i$, $i = 1,\dots, k - 1$, $x_i = s_{i+1}$, $i = k,\dots, n - 1$, and $x_n = b$. Since M has property (WC-2) there exists a function v_k, $v_k \neq 0$, such that $(-1)^{i+1} v_k(x) \geqslant 0$, $x_{i-1} \leqslant x \leqslant x_i$, $i = 1,\dots, n$. Since v_k may be represented as $\sum \lambda_i u_i$ it is readily seen that $v_k = \lambda_k u_k$ for some $\lambda_k > 0$. Furthermore, u_k may be represented as

$$u_k(x) = \frac{1}{\det[\phi_i(s_j)]} \begin{vmatrix} \phi_1(x) & \cdots & \phi_n(x) \\ \phi_1(s_1) & \cdots & \phi_n(s_1) \\ & \cdots & \\ \phi_1(s_{k-1}) & \cdots & \phi_n(s_{k-1}) \\ \phi_1(s_{k+1}) & \cdots & \phi_n(s_{k+1}) \\ & \cdots & \\ \phi_1(s_n) & \cdots & \phi_n(s_n) \end{vmatrix} \tag{1}$$

Now let $a \leqslant t_1 < \cdots < t_n \leqslant b$ be points such that $\det(\phi_i(t_j)) \neq 0$. Suppose, k is such that $s_k \notin \{t_1, ..., t_n\}$. Since $u_k \neq 0$ and $\det[\phi_i(t_j)] \neq 0$, there exists a t_m such that $u_k(t_m) \neq 0$. It follows from formula (1) that $t_m \notin \{s_i\}_{i=1/i\neq k}^n$. Let $\{r_i\}_{i=1}^n = \{s_i\}_{i=1/i\neq k}^n \cup \{t_m\}$, so that $r_i < r_{i+1}$, $i = 1,..., n-1$. Then it follows from formula (1) and from the relation $u_k = (1/\lambda_k)\, v_k$ that

$$\det[\phi_i(s_j)] \det[\phi_i(r_j)] > 0.$$

By a repeated application of this argument we find

$$\det[\phi_i(s_j)] \det[\phi_i(t_j)] > 0,$$

which clearly completes the proof.

(WC) $\Rightarrow$ (WC-1). Let $\phi_1,...,\phi_n$ be a basis for M. Let $a < s_1 < \cdots < s_n < b$ be points such that $\det[\phi_i(s_j)] \neq 0$. Define (uniquely) the functions $u_j \in M$, $j = 1,..., n$, by $u_j(s_i) = (-1)^{i+1}\,\delta_{ij}$, $i = 1,..., n$. For fixed k, let $x_0 = a$, $x_i = s_i$, $i = 1,..., k-1$, $x_i = s_{i+1}$, $i = k,..., n-1$, and $x_n = b$. We now show that u_k has the property: $(-1)^{i+1}\, u_k(x) \geqslant 0$, $x_{i-1} \leqslant x \leqslant x_i$, $i = 1,..., n$. Otherwise, there exists an i_0 and points y, z such that $x_{i_0-1} < y < z < x_{i_0}$ and such that $u_k(y)\, u_k(z) < 0$. Let $h = \sum_{i<k} u_i - \sum_{k<i} u_i$. Then $h(x_i) = (-1)^{i+1}$ $i = 1,..., n-1$. Consider the function $g_\lambda = u_k + \lambda h$. Then $g_\lambda(x_i) = \lambda(-1)^{i+1}$ $i = 1,..., n-1$, $g_\lambda(y) = u_k(y) + \lambda h(y)$, and $g_\lambda(z) = u_k(z) + \lambda h(z)$. Thus, it is readily seen that for sufficiently small λ of appropriate sign, the function g_λ changes sign $n + 1$ times on the set $\{x_1, ..., x_{i_0-1}, y, z, x_{i_0}, ..., x_{n-1}\}$. This contradicts our assumption that M has property (WC); hence, u_k, $k = 1, 2,..., n$, has the desired properties. By virtue of this we can proceed exactly as in the Proof of (WC-2) $\Rightarrow$ (WC-1) above, to show that if $a \leqslant t_1 < \cdots < t_n \leqslant b$ is any set of points such that $\det[\phi_i(t_j)] \neq 0$, then

$$\det[\phi_i(s_j)] \det[\phi_i(t_j)] > 0.$$

This proves that M has property (WC-1). This concludes the proof of the equivalence of properties (WC), (WC-1) and (WC-2).

Proof of the Theorem. We first show that if M has property (WC) then for each $f \in C[a, b]$ there exists at least one $g \in V_f(M)$ such that the error $e = f - g$ equioscillates. Let $\phi_1, \ldots, \phi_n$ be a basis for M, and for each $\sigma > 0$ let $M_\sigma = \operatorname{span}\{\psi_1{}^\sigma, \ldots, \psi_n{}^\sigma\}$ be as in the Lemma. For each integer $k \geqslant 3/(b-a)$ denote by I_k the interval $[a + 1/k, b - 1/k]$, and define the seminorm $\| \cdot \|_k$ on $C[a, b]$

$$\| h \|_k = \max\{| h(x)| : x \in I_k\}.$$

Let f be an arbitrary element of $C[a, b]$ which does not belong to M. For each $\sigma > 0$ let $g_{\sigma k}$ be the element of M_σ satisfying

$$\|f - g_{\sigma k}\|_k \leqslant \|f - h\|_k \qquad \text{for all } h \in M_\sigma . \tag{2}$$

The uniqueness of $g_{\sigma k}$ follows from the fact that the restriction of M_σ to I_k also has property (C). This, furthermore, implies that the error $e_{\sigma k} = f - g_{\sigma k}$ equioscillates on I_k, i.e., there exist points

$$a + 1/k \leqslant x_1^{\sigma k} < \cdots < x_{n+1}^{\sigma k} \leqslant b - 1/k$$

such that

$$| e_{\sigma k}(x_i^{\sigma k})| = \| e_{\sigma k} \|_k , \qquad i = 1, \ldots, n + 1,$$

and

$$e_{\sigma k}(x_i^{\sigma k})\, e_{\sigma k}(x_{i+1}^{\sigma k}) < 0, \qquad i = 1, \ldots, n. \tag{3}$$

Here we have assumed that $f \notin M_\sigma$. For sufficiently small σ and sufficiently large k this follows from the fact that f does not belong to M.

We represent $g_{\sigma k}$ as $\sum_{i=1}^n \lambda_i^{\sigma k} \psi_i{}^\sigma$. From (2), $\| g_{\sigma k} \|_k \leqslant 2 \| f_k \|_k$ for every $\sigma > 0$. Hence, choosing k so large that the ϕ_i are linearly independent on I_k one has that the coefficients $\lambda_i^{\sigma k}$ are uniformly bounded in σ and in i. By virtue of this we may choose a sequence σ_ν, $\sigma_\nu \to 0$, and numbers $\lambda_i{}^k, \ldots, \lambda_n{}^k$ so that $\lambda_i^{\sigma_\nu k} \to \lambda_i{}^k$ as $\nu \to \infty$, $i = 1, \ldots, n$. Since $\psi_i{}^\sigma \to \phi_i$, uniformly on I_k, as $\sigma \to 0$, it follows that $g_{\sigma_\nu k} \to g_k = \sum_{i=1}^n \lambda_i{}^k \phi_i$, uniformly on I_k. We may choose a subsequence of σ_ν, again denoted by σ_ν, and numbers $x_1{}^k, \ldots, x_{n+1}^k$ so that $x_i^{\sigma_\nu k} \to x_i{}^k$ as $\nu \to \infty$, $i = 1, \ldots, n + 1$. If we choose k so large that f is linearly independent of $\phi_1, \ldots, \phi_n$ on I_k then it follows from the uniform convergence of the $g_{\sigma_\nu k}$ and from (3) that if $e_k = f - g_k$ then

$$| e_k(x_i{}^k)| = \| e_k \|_k , \quad i = 1, \ldots, n + 1, \quad e_k(x_i{}^k)\, e_k(x_{i+1}^k) < 0, \quad i = 1, \ldots, n. \tag{4}$$

In particular, (4) implies that $x_i{}^k < x_{i+1}^k$, $i = 1, \ldots, n$. Since M has property (WC) it also follows from (4) that

$$\|f - g_k\|_k \leqslant \|f - h\|_k \qquad \text{for all} \quad h \in M. \tag{5}$$

For if $\|f - h\|_k < \|f - g_k\|_k$ for some $h \in M$ then by virtue of (4):

$$\{[h(x_i^k) - g_k(x_i^k)][h(x_{i+1}^k) - g_k(x_{i+1}^k)]\} < 0, \qquad \text{for } i = 1,..., n,$$

which contradicts the fact that M has property (WC). From (5), for fixed k_0, we have

$$\| g_k \|_{k_0} \leqslant \| g_k \|_k \leqslant 2 \|f\|_k \leqslant 2 \|f\|, \qquad \text{for all } k \geqslant k_0 . \tag{6}$$

Hence, if k_0 is chosen so large that $\phi_1 ,..., \phi_n$ are linearly independent on I_{k_0} it follows from (6) that the coefficients λ_i^k of $g_k = \sum_{i-1}^n \lambda_i^k \phi_i$ are uniformly bounded in i and in k. Hence, we can choose a subsequence s_k of 1, 2, 3,... and numbers $\lambda_1 ,..., \lambda_n$ so that $\lambda_i^{s_k} \to \lambda_i$, as $k \to \infty$, $i = 1,..., n$, and

$$g_{s_k} \to g = \sum_{i=1}^n \lambda_i \phi_i , \qquad \text{as } k \to \infty, \text{ uniformly on } [a, b]. \tag{7}$$

We can choose a further subsequence, again denoted by s_k, and points $x_1 ,..., x_{n+1}$ so that $x_i^{s_k} \to x_i$, as $k \to \infty$, $i = 1,..., n + 1$. Letting $e = f - g$, (4) and (7) imply

$$| e(x_i)| = \| e \|, \qquad i = 1,..., n + 1, \qquad e(x_i)\, e(x_{i+1}) < 0, \qquad i = 1,..., n \tag{8}$$

The fact that $g \in V_f(M)$ follows from (5) and (7). Also, as above, it follows from (8) and property (WC) of M. Moreover, by (8), $e = f - g$ has the desired equioscillation property and the sufficiency is proven.

Finally, we show that if M has the property that for each $f \in C[a, b]$ there exists at least one $g \in V_f(M)$ such that the error $e = f - g$ equioscillates then M has property (WC). Given arbitrary points

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

we now show that there exists a $g \in M$ such that $g \not\equiv 0$ and such that $(-1)^{i+1} g(x) \geqslant 0$, $x_{i-1} < x < x_i$, $i = 1,..., n$. For each $\epsilon > 0$ sufficiently small define the function f_ϵ by

$$f_\epsilon(x) = \begin{cases} 0, \quad x = x_{i-1}, \quad x_i, \\ (-1)^{i+1}, \quad x_{i-1} + \epsilon \leqslant x \leqslant x_i - \epsilon, \\ \text{linear elsewhere on} \quad [x_{i-1}, x_i], \end{cases}$$

$$i = 1, 2,..., n.$$

By assumption, for each ϵ there exists a $g_\epsilon \in V_{f_\epsilon}(M)$ so that $e_\epsilon = f_\epsilon - g_\epsilon$ equioscillates. In particular this means that $\|f_\epsilon - g_\epsilon\| \leqslant \|f_\epsilon - 0\| = 1$. Moreover $g_\epsilon \not\equiv 0$ because $f_\epsilon - 0 = f_\epsilon$ does not equioscillate since it changes sign only $n-1$ times. Now $\|f_\epsilon - g_\epsilon\| \leqslant \|f_\epsilon\| = 1$ implies that $(-1)^{i+1} g_\epsilon(x) \geqslant 0$ for $x_{i-1} + \epsilon \leqslant x \leqslant x_i - \epsilon$, $i = 1,..., n$. Since $\|g_\epsilon\| \neq 0$ the functions $h_\epsilon = g_\epsilon / \|g_\epsilon\|$ have the same property. Choosing a convergent subsequence h_{ϵ_ν} such that $h_{\epsilon_\nu} \to g$ as $\nu \to \infty$ it is easily observed that g has the desired properties: $(-1)^{i+1} g(x) \geqslant 0$, $x_{i-1} < x < x_i$, $i = 1,..., n$, and $g \not\equiv 0$. This concludes the proof.

COROLLARY. *Let M be a finite dimensional linear subspace of $C[a, b]$. For each $f \in C[a, b]$ there exists a $g \in V_f(M)$ such that the error $e = f - g$ equioscillates, if and only if, this is true for each f having the additional property that $V_f(M)$ is a singleton.*

Proof. In one direction this is trivial. In the other direction it is sufficient to show that M has property (WC). Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be arbitrary points. Let f_ϵ be as in the proof of the Theorem. Either $V_{f_\epsilon}(M)$ is not a singleton and there exists an $h_\epsilon \in V_{f_\epsilon}(M)$ such that $h_\epsilon \not\equiv 0$ or $V_{f_\epsilon}(M) = \{g_\epsilon\}$ and $f_\epsilon - g_\epsilon$ equioscillates. Moreover, it follows that $(-1)^{i+1} g_\epsilon(x) > 0$, $x_{i-1} + \epsilon \leqslant x \leqslant x_i - \epsilon$, $i = 1,..., n$ and $(-1)^{i+1} h_\epsilon(x) \geqslant 0$, $x_{i-1} + \epsilon \leqslant x \leqslant x_i - \epsilon$, $i = 1,..., n$. In either case, we proceed as in the proof of the Theorem to show that there exists a $g \in M$ such that $g \neq 0$ and $(-1)^{i+1} g(x) \geqslant 0$, $x_{i-1} \leqslant x \leqslant x_i$, $i = 1,..., n$. This concludes the proof.

3. EXAMPLES AND COMMENTS

We begin with a simple example of a subspace M which has property (WC).

Piecewise linear functions. Let $[a, b]$ be a finite interval. Choose knots $a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = b$. Let M be the linear subspace of continuous, piecewise linear functions with (possible) corners at the knots, i.e., let $M = \{g : g \in C[a, b], g \text{ linear on } [\xi_{i-1}, \xi_i], i = 1,..., n\}$. Then M has property (WC), for the dimension of M is $n + 1$, and if there exists a $g \in M$ and points $a \leqslant x_1 < \cdots < x_{n+2} \leqslant b$ such that $g(x_i)\, g(x_{i+1}) < 0$, $i = 1,..., n + 1$, then there exist points $y_1, ..., y_{n+1}$ such that $x_i < y_i < x_{i+1}$, $i = 1,..., n + 1$, and such that $g'(y_i)\, g'(y_{i+1}) < 0$, $i = 1,..., n$. It follows that there exist i_0 and j_0 such that $y_{i_0}, y_{i_0+1} \in [\xi_{j_0}, \xi_{j_0+1}]$, which contradicts the fact that g is linear on $[\xi_{j_0}, \xi_{j_0+1}]$.

This example can be generalized as follows.

Spline polynomials. For any positive integer k we define the function $(x)_+^k$ by

$$(x)_+^k = \begin{cases} x^k, & x \geqslant 0, \\ 0, & x \leqslant 0. \end{cases}$$

Choose knots $-1 < \xi_1 < \cdots < \xi_n < 1$, and let k be an arbitrary positive integer. It has been shown by Karlin and Studden [1], page 18, that the linear subspace M spanned by

$$x^k, x^{k-1}, \ldots, x, 1, (x - \xi_1)_+^{\ k}, \ldots, (x - \xi_n)_+^{\ k},$$

has property (WC-1) on the interval $[-1, +1]$.

Construction of best approximation. If M has property (C) then for each $f \in C[a, b]$ the well known Remez algorithm can be used (at least in theory) to construct a sequence $\{g_m\}$ in M which converges uniformly to g, where $\{g\} = V_f(M)$. This is based on the equioscillation property of $e = f - g$ and on the fact that $V_f(M)$ is a singleton. No general algorithm exists for the case where M has property (WC). It is hoped that by virtue of our theorem above something analogous to the Remez algorithm can be developed. In this regard we note *that if M has property* (WC) *and if we have found a $g \in M$ such that $e = f - g$ equioscillates then $g \in V_f(M)$.* (This was verified and used in the proof of the Theorem.)

Geometric considerations. If M is as arbitrary finite dimensional linear subspace of $C[a, b]$ then for each f the set $V_f(M)$ is convex and compact. It follows directly from a result of Brosowski ([2], Section 4, Satz 6), that if g belongs to the relative interior of $V_f(M)$ and if $e = f - g$ equioscillates then $e = f - h$ equioscillates for every $h \in V_f(M)$. Hence, in general we expect to find those g for which $e = f - g$ equioscillates on the boundary of $V_f(M)$. It can also be shown that if M has property (WC) then there exists an extreme point g of $V_f(M)$ for which $e = f - g$ equioscillates. However, this is not true for every extreme point. Further geometric characterizations would be interesting.

REFERENCES

1. S. KARLIN AND W. J. STUDDEN, "Tchebycheff Systems: with Application in Analysis and Statistics," John Wiley and Sons, Inc., New York, 1966.
2. B. BROSOWSKI, Über Tschebyscheffsche Approximationen mit linearen Nebenbedingungen, *Math. Z.* **88** (1965), 105–128.